A Laboratory on Cubic Polynomials

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Most ***Crux*** readers will be familiar with quadratic polynomials, and know how to solve the corresponding equations by completing the square or using the quadratic formula. Far fewer people know how to solve a cubic equation! In this lab, we will study polynomials of third degree $P(x) = ax^3 + bx^2 + cx + d$, corresponding to the cubic equation

$$ax^3 + bx^2 + cx + d = 0 \ ,$$

(We will always assume the coefficients of the polynomial to be real, though we will consider roots which are not.)

We can reduce the polynomial to monic form (in which the leading coefficient equals 1) in various ways. The simplest way is, of course, to divide the entire polynomial by a. But we can also do it by a change of variables : letting $x = (t/a)$ and multiplying by a^2, we obtain the monic equation

$$t^3 + bt^2 + act + a^2 d = 0 \ .$$

Such a reduction is useful when we want to keep the coefficients as integers! In the particular case where $d = 1$, obviously $x = 0$ is not a root, and we can also reduce the equation to monic form by substituting $x = (1/t)$.

In what follows, we suppose that the cubic equation has – somehow or other – the form

$$x^3 + rx^2 + px + q = 0 \qquad \text{for } p, q, r \text{ real.} \tag{1}$$

Problem 1 *Prove that the equation (1) always has at least one real root.*

Hint : Show that there exists a pair of numbers m, M with $P(M)P(m) < 0$. Then use the Intermediate Value Theorem.

Problem 2 *Prove that, for any $a \in R$, $f(x) = (x - a) \cdot g(x) + f(a)$, where $g(x)$ is a quadratic polynomial.*

Hint : Which quadratic polynomial?

Let $x = a$ be a root of $f(x)$. Then $f(x) = (x - a) \cdot g(x)$, where $g(x)$ is a quadratic polynomial. If $g(a) \neq 0$ we shall say that $x = a$ is a *simple root* of the equation $f(x) = 0$; otherwise it is a *multiple root.* If $g(a) = 0$, we have two possibilities : either $g(x) = (x-a)^2$ or $g(x) = (x-a) \cdot (x-b)$ for $b \neq a$. In the first case the root a has multiplicity 3 and $f(x) = (x - a)^3$. In the second case, it has multiplicity 2 and $f(x) = (x - a)^2 \cdot (x - b)$.

Problem 3 *Prove that for any cubic equation of form (1) one of the following must hold; and find an example of each case.*

1. *$f(x) = 0$ has one simple real root and no other real roots*
2. *$f(x) = 0$ has three simple real roots*

3. *$f(x) = 0$ has one real root of multiplicity 2 and one simple real root*
4. *$f(x) = 0$ has one real root of multiplicity 3.*

What conditions on p, q, r force each case ?

If the equation $f(x) = 0$ has all of its roots real, then one of (2-4) must be the case. Only the case of one simple real root is eliminated.

Problem 4 *Prove that the three real numbers x_1, x_2, x_3 (some or all of which could be equal) are roots of equation (1) if and only if they satisfy the following three conditions :*

$$\begin{cases} x_1 + x_2 + x_3 = -r \\ x_1x_2 + x_2x_3 + x_3x_1 = p \\ x_1x_2x_3 = -q \end{cases} \tag{2}$$

This is the cubic case of Viète's theorem *and the above system of equations is called* Viète's system.

Before we go on, we note that equation 1 can be reduced by the further substitution $x = y - \frac{r}{3}$ to the form

$$y^3 + b \cdot y + c = 0 \tag{3}$$

with no quadratic term.

Problem 5 *Derive formulae for b, c in terms of p, q, r.*

Now we consider this equation in more detail.

Problem 6 *Show that if $b > 0$, then equation (3) has a unique real root, which if we hold b fixed (say $b = 1$) can be considered as a function $y(c)$ of the other coefficient. Show that if, furthermore, $c > 0$, then $y(c) < 0$.*

Problem 7 *Show that $y(c)$ as defined above is monotone decreasing, continuous, and twice differentiable on $(-\infty, \infty)$. Find the derivative dy/dc.*

Problem 8 *Let $y(c)$ be as defined above.*

a) *Find the second derivative d^2y/dc^2.*
b) *Show that $y(c)$ is concave down for $c < 0$, concave up for $c > 0$, and has a point of inflection at $c = 0$. (You can prove this part without using derivatives.)*

We now return to the question of computing the roots of equation (3). The case $b = 0$ is fairly trivial (what are the roots in this case ?), so we concentrate on the remaining cases.

Case I : $b > 0$. Make the substitution

$$y = \sqrt{\frac{b}{3} \cdot \left(t - \frac{1}{t}\right)}$$

in equation (3) and manipulate it to obtain a quadratic equation in t^3.

- Is it always soluble within the real number system ?
- Is it always equivalent to the original equation in the sense of having the same set of roots ?
- If this calculation has introduced *extraneous* roots that are not roots of the original equation, how can they be removed ?

Case II : $b < 0$. Make the substitution

$$y = \sqrt{\frac{b}{3} \cdot \left(t + \frac{1}{t}\right)}$$

in equation (3).

- In this case, it is possible that the resulting quadratic equation has no roots - when does this happen ?
- What conditions on b and c will guarantee roots ?
- The quadratic equation may have two roots - what do we do with them ?
- Are there extraneous roots ?
- Have we found all the roots of equation (1), and if not how do we find the rest ?

The next approach uses a trigonometric transformation.

Problem 9 *Consider the cubic equation*

$$4t^3 - 3t = d \tag{4}$$

where $|d| \leq 1$*. Set* $d = \cos(\alpha)$ *and use the substitution* $t = \cos(\phi)$ *to show that*

$$\left\{ t_0 := \cos(\frac{\alpha}{3}),\ t_1 := \cos(\frac{\alpha + 2\pi}{3}),\ t_2 := \cos(\frac{\alpha + 4\pi}{3}) \right\}$$

is the full set of real roots of (4). How does the multiplicity of the roots depend on d *?*

Problem 10 *Suppose that* $|d| > 1$ *in equation (4). Prove that*

$$t_0 := \frac{\sqrt[3]{d + \sqrt{d^2 - 1}} - \sqrt[3]{d - \sqrt{d^2 - 1}}}{2}$$

is the unique real root.

Hint : prove that $|t_0| \geq 1$.

Problem 11 *Consider the cubic equation*

$$4t^3 + 3t = d \tag{5}$$

Prove that

$$t_0 := \frac{\sqrt[3]{\sqrt{d^2-1}+d} - \sqrt[3]{\sqrt{d^2-1}-d}}{2}$$

is its unique real root.

Problem 12 *Show that the equation* $y^3 + by + c = d$ *can be put into the form (4) or (5) by the substitution* $y = 2\sqrt{|b|/3}t$*, depending on the sign of* b*.*

Problem 13 *Use Problem (12) to formulate the results of problems (9-11) for the equation* $y^3 + by + c = 0$*. How does the value* $D := (q/2)^2 + (p/3)^3$ *help classify the possible cases ?*

We now consider cubic equations with real coefficients but with *complex* roots. For convenience, we will write our original equation in the form

$$x^3 - rx^2 + px - q = 0\ . \tag{6}$$

Problem 14 *Prove that this equation always has three solutions in the set of complex numbers, counting each root with its correct multiplicity.*

We will introduce a *discriminant* for cubic equations. Readers will recall that for a reduced *quadratic* $x^2 + bx + c$ with roots x_1, x_2, the discriminant is defined to be

$$\Delta := b^2 - 4c = (x_1 + x_2)^2 - 4x_1x_2 = (x_1 - x_2)^2\ .$$

When this is positive, the quadratic has two distinct real roots. When it is zero, it has a double root, and when it is negative, a complex-conjugate pair.

For a reduced *cubic* polynomial, the discriminant is defined to be

$$\Delta := (x_1 - x_2)^2(x_2 - x_3)^2(x_3 - x_1)^2\ .$$

This has similar properties.

Problem 15 *Prove the following statements :*

1. $\Delta > 0 \Leftrightarrow$ *all three roots are distinct and real ;*

2. $\Delta = 0 \Leftrightarrow$ *at least two roots are equal and all are real ;*

3. $\Delta < 0 \Leftrightarrow$ *one root is real and the others are complex conjugates*

Problem 16 *Suppose the cubic polynomial to be in the form (3). Show that* $\Delta = -4a^3 - 27b^2$*.*

The discriminant Δ doesn't give us complete information about the nature of the roots : when $\Delta = 0$ we don't know whether we have one double root and one single root, or one triple root. We can determine this using

$$\Delta_1 := (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2\ .$$

Problem 17 *Suppose that $\Delta = 0$. Show that there is a triple root if and only if $\Delta_1 = 0$.*

Problem 18 *For a cubic polynomial in the form (6), prove that*

$$\Delta = r^2p^2 - 4r^3q - 4p^3 - 27q^2 + 18rpq \, .$$

Problem 19 *For a cubic polynomial in the form (6), prove that*

$$\Delta_1 = 2(r^2 - 3p) \, .$$

Problem 20 *For a cubic polynomial in the form (6), prove that necessary and sufficient conditions for all roots to be real and positive are*

$$r, p, q, \Delta \geq 0 \, .$$

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The article is adapted with permission from an article by Arkady Alt in Delta *(Nov. 1994).*

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Math Quotes

In the fall of 1972 President Nixon announced that the rate of increase of inflation was decreasing. This was the first time a sitting president used the third derivative to advance his case for reelection.

From "Mathematics Is an Edifice, Not a Toolbox", by Hugo Rossi, Notices of the AMS, v. 43, no. 10, October 1996.